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Nilpotency and Theory of L-Subgroups of an L-Group

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Abstract In this paper, the notion of commutator is modified and extended to L-setting. Also, the notion of descending central series is introduced which is used to formulate the important notion of nilpotent L-subgroup of an L-group. Moreover, the level subset characterization for the notion of nilpotent L-subgroup is provided.

Keywords L-Algebra · L-Subgroup · Generated L-subgroup · Normal L-subgroup · Nilpotent L-subgroup

1. Introduction

Ever since its inception, the theory of fuzzy sets has been applied to various branches of information theory, computer science and engineering, whereas group theory has been effectively applied to different fields of science and technology. Keeping in view the applications of these disciplines, Rosenfeld [18] laid the foundation of the theory of fuzzy groups. On the other hand, lattice theory, in fact, has found many significant applications in different areas of human knowledge systems. This theory possesses potential to unify diverse concepts and has been applied effectively in

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various branches of information sciences like computational intelligence, neural networks, pattern recognition, mathematical morphology etc. In this paper, we amalgamate all these disciplines and pursue lattice valued fuzzy subgroups which are called L-subgroups. Hopefully, such studies may not only reveal deeper structure of fuzzy subgroups but also develop new methods to applications.

Gupta and Sarma in [9] extended the important notion of commutator subgroup to fuzzy setting in 1996. This was in turn used to define the notion of descending central chain which is utilized to formulate the concept of nilpotent fuzzy subgroup of a group [10]. Besides, Kim defined an ascending series of fuzzy subgroups of a group to introduce his concept of nilpotent fuzzy subgroups [12]. The former approach was found to be more consistent with the other existing notions of fuzzy group theory. It is worth noticing that all the above mentioned studies have been carried out within the framework where the underlying group is an ordinary group.

Recall that the idea of a normal fuzzy subgroup of a fuzzy group was introduced by Wu in his paper [21]. However, the authors in [10] did not utilize these ideas. They used the normality of the members of the descending central series in the whole group G in the sense of Liu [13] rather than using the normality of the members of the descending central series in the preceding ones in the sense of Wu.

After some initial attempts [13-15], we have taken a major step forward towards the development of the theory on fuzzy subgroups in a fuzzy group by introducing the concept of characteristic fuzzy subgroup of a fuzzy group and established its relationship with certain other notions [4]. In the present paper, we proceed to develop a similar theory of L-subgroups of an L-group. We provide the definition of nilpotent L-subgroup of an L-group and investigate its properties. Before doing this, we modify and extend the notion of commutator to L-setting. Many results involving commutator L-subgroups are established with the help of certain manipulations with the infimums of the given L-subgroups. The results are not only extended to L-setting but also the definition of commutator is changed very diligently in such a way that it has far reaching consequences. An L-subgroup of an L-group is said to be nilpotent if its descending central chain terminates finitely at the trivial L-subgroup of the given L-subgroup. The notion of trivial L-subgroup of an L-subgroup used in our definition differs from that of trivial fuzzy group used in [10], which is an ordinary fuzzy point. In order to justify the suitability of this extension, we establish the level subset characterization of nilpotent L-subgroup, which is much more a general result than that of characterization of nilpotent subgroups by the characteristic function provided in [10].

2. Preliminaries

Throughout this paper, $L = \langle L, \leq, \vee, \wedge \rangle$ denotes a completely distributive lattice where '$\leq$' denotes the partial ordering of L, the join (sup) and the meet (inf) of the elements of L are denoted by '$\vee$' and '$\wedge$' respectively. Also, we write 1 and 0 for maximal and minimal elements of L respectively. In this section, we first introduce some basic definitions and results which are used in the sequel. For details, we refer to [1-2, 7-8, 16-19, 22].

An L-subset of X is a function from X into L. The set of L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) \mid x \in X\}$ is called the image of μ and is denoted by $\operatorname{Im}\mu$ and the tip of μ is defined as $\bigvee_{x\in X}\{\mu(x)\}$. Also, $\bigwedge_{x\in X}\{\mu(x)\}$ is known as the tail of μ. If $\mu, \nu \in L^X$, then we say that μ is contained in ν if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted by $\mu \subseteq \nu$. For a family $\{\mu_i \mid i \in I\}$ of L-subsets in X, where I is a nonempty index set, the union $\bigcup_{i\in I}\mu_i$ and the intersection $\bigcap_{i\in I}\mu_i$ of a family $\{\mu_i \mid i \in I\}$ are defined as usual. For the definitions of set product $\mu \circ \nu$, level subset μ_a, strong level subset $\mu_a^>$, L-subgroup and L-normal subgroup, we refer to [7,16,17].

Proposition 2.1 *Let $\mu, \nu \in L^X$. Then*

(i) *if $\mu \subseteq \nu$, then $\mu_a \subseteq \nu_a$ for each $a \in L$,*

(ii) *if $\mu_a \subseteq \nu_a$ for each $a \in \operatorname{Im}\mu$, then $\mu \subseteq \nu$.*

A similar formulation for the notion of strong level subset is as follows:

Proposition 2.2 *Let $\mu, \nu \in L^X$. Then*

(i) *if $\mu \subseteq \nu$, then $\mu_a^> \subseteq \nu_a^>$ for each $a \in L \sim \{1\}$,*

(ii) *if $\mu_a^> \subseteq \nu_a^>$ for each $a \in \operatorname{Im}\nu$, then $\mu \subseteq \nu$ provided L is a chain.*

Throughout this paper, G denotes an ordinary group with the identity element 'e' and I denotes a nonempty indexing set. If $A \subseteq X$, then by 1_A we denote the characteristic function of A in X. The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element 'e' of G. Moreover, $\mu \in L(G)$ is said to be a proper L-subgroup of G if μ is non-constant. It is well known that the intersection of an arbitrary family of L-subgroups of a group is an L-subgroup.

For an L-subset μ of G, the L-subgroup of G generated by μ is defined as the smallest L-subgroup of G which contains μ which is denoted by $\langle\mu\rangle$.

If $\mu, \eta \in L^X$ and $\eta \subseteq \mu$, then we say that η is an L-subset of μ. We denote the set of L-subsets of μ by $L^{(\mu,G)}$. If $\mu, \eta \in L(G)$ and $\eta \subseteq \mu$, then we say that η is an L-subgroup of μ. Further if η is non-constant and $\eta \neq \mu$, then η is said to be a proper L-subgroup of μ. Also, η is said to be a trivial L-subgroup of μ if its chain of level subgroups contains only $\{e\}$ and G.

Let η be an L-subgroup of μ. Then we define the following L-subgroup of μ contained in η, denoted by $\eta_{t_0}^{a_0}$, as follows:

$$\eta_{t_0}^{a_0}(y) = \begin{cases} a_0, & \text{if } y = e, \\ t_0, & \text{if } y \neq e, \end{cases}$$

where $a_0 = \eta(e)$ and $t_0 = \inf \eta$. Here $\eta_{t_0}^{a_0}$, a trivial L-subgroup of μ, is called the trivial L-subgroup of η.

For convenience, we use the notation $L(\mu, G)$ for the set of L-subgroups of μ. This terminology is adopted in view of the fact that in this paper, we mainly discuss L-substructures in an L-subgroup μ of G instead of L-substructures in an ordinary group G. Hence onwards μ denotes an L-subgroup of G and we shall call the parent L-subgroup μ simply an L-group. Below we provide the level subset [17] and strong level subset characterizations of an L-subgroup in the setting of $L(\mu, G)$.

Theorem 2.1 *Let $\eta \in L^{(\mu,G)}$ with tip a_0. Then*

(i) *$\eta \in L(\mu, G)$ if and only if η_a is a subgroup of μ_a for each $a \leq a_0$.*

(ii) *$\eta \in L(\mu, G)$ if and only if $\eta_a^>$ is a subgroup of $\mu_a^>$ for each $a < a_0$ provided L is a chain.*

Here note that for every subgroup H of G, 1_H is an L-subgroup of the L-group 1_G.
If $\eta \in L^{(\mu,G)}$ and $\langle\eta\rangle_\mu$ denotes the L-subgroup of μ generated by η, then it can be easily verified that $\langle\eta\rangle_\mu = \langle\eta\rangle$.

Definition 2.1 [21] *Let $\eta \in L(\mu, G)$. Then η is said to be a normal L-subgroup of μ if for all $x, y \in G$,*

$$\eta(yxy^{-1}) \geq \eta(x) \wedge \mu(y).$$

The set of normal L-subgroups of μ is denoted by $\mathrm{NL}(\mu, G)$.

The following results which appear for the first time [1] were restricted to fuzzy group theory. Below we state the generalizations of these results in the setting of $L(\mu, G)$.

Theorem 2.2 *Let $\eta \in \mathrm{NL}(\mu, G)$ and $\theta \in L(\mu, G)$. Then $\eta \circ \theta \in L(\mu, G)$.*

Theorem 2.3 *Let $\eta, \theta \in \mathrm{NL}(\mu, G)$. Then $\eta \circ \theta \in \mathrm{NL}(\mu, G)$.*

Definition 2.2 *Let $\eta \in L^{(\mu,G)}$. Then η is said to have sup-property if for each nonempty subset A of G, there exists $a_0 \in A$ such that $\sup_{a \in A}\{\eta(a)\} = \eta(a_0)$.*

If $x, y \in G$, then their commutator $xyx^{-1}y^{-1}$ and conjugate $y^{-1}xy$ are denoted by $[x, y]$ and x^y respectively.

3. Commutator L-Subgroup

The notion of a commutator and that of a commutator subgroup play a very significant role in the classical group theory. In particular, the concept of nilpotent group is formulated using the notion of commutator subgroups. In the fuzzy setting, the commutator of two fuzzy subsets has been introduced in [9]. In this section, we modify and extend the notion of commutator to L-setting by using the notion of infimums. It is demonstrated in this section that the notion of infimums plays a vital role while establishing several results. In fact, a development of the theory in L-subgroups in an L-group cannot be perceived without an effective role of infimums.

Definition 3.1 *Let $\eta, \theta \in L^{(\mu,G)}$. Then the commutator of η and θ is an L-subset (η, θ) of G defined as follows:*

$$(\eta, \theta)(x) = \begin{cases} \vee\{\eta(y) \wedge \theta(z)\}, & \text{if } x = [y, z] \text{ for some } y, z \in G, \\ \inf \eta \wedge \inf \theta, & \text{if } x \neq [y, z] \text{ for any } y, z \in G. \end{cases}$$

The commutator L-subgroup of $\eta, \theta \in L^{(\mu,G)}$ is defined as the L-subgroup of G generated by (η, θ). It is denoted by $[\eta, \theta]$. Clearly, $\inf(\eta, \theta) = \inf \eta \wedge \inf \theta$ and $[\eta, \theta] \in L(\mu, G)$.

The following example of a commutator illustrates our view point:

Example 3.1 Let S_4 be the group of all permutation on the set $\{1, 2, 3, 4\}$ with the identity element 'e' and A_4 be the corresponding alternating group. Let

$$D_4 = \langle(24), (1234)\rangle = \{e, (12)(34), (13)(24), (14)(23), (13), (24), (1234), (1432)\}$$

be the dihedral subgroup of S_4 with centre $C = \{e, (13)(14)\}$ and

$$V_4 = \{e, (12)(34)(13)(24), (14)(23)\} \subseteq D_4(A_4)$$

denote the Klein-4 subgroup.

Define the following L-subsets μ, η and θ of S_4 as follows:

$$\begin{aligned}
&\mu(V_4) = 1 \quad \text{and} \quad \mu(S_4 \sim V_4) = \frac{1}{2};\\
&\eta(C) = 1, \quad \eta(\langle(1234)\rangle \sim C) = \frac{1}{2},\\
&\eta(D_4 \sim \langle(1234)\rangle) = \frac{1}{4} \quad \text{and} \quad \eta(S_4 \sim D_4) = \frac{1}{6};\\
&\theta(V_4) = 1, \quad \theta(A_4 \sim V_4) = \frac{1}{3} \quad \text{and} \quad \theta(S_4 \sim D_4) = \frac{1}{5}.
\end{aligned}$$

Here $A \sim B$ denotes usual set difference. Evidently, $\mu \in L(S_4)$ and $\eta, \theta \in L(\mu, S_4)$.

Further, it is easy to verify that the commutator (η, θ) is given by

$$\begin{aligned}
&(\eta, \theta)(e) = 1, \quad (\eta, \theta)(C \sim e) = \frac{1}{2}, \quad (\eta, \theta)(A_4 \sim C) = \frac{1}{3}, \quad \text{and}\\
&(\eta, \theta)(S_4 \sim A_4) = \frac{1}{6}.
\end{aligned}$$

Moreover note that $(\eta, \theta) = [\eta, \theta] \in L(\mu, S_4)$ and $\inf(\eta, \theta) = \inf \eta \wedge \inf \theta$.

Recall that the construction of a fuzzy subgroup generated by a given fuzzy subset of a group has been discussed in detail in various papers [2-3, 6, 20]. Here we provide a similar construction for generating an L-subgroup by a given L-subset of an L-group. The proof of the following result can be found in [5].

Theorem 3.1 *Let $\eta \in L^{(\mu,G)}$. If $a_0 = \bigvee_{x \in G} \{\eta(x)\}$, then define an L-subset $\hat{\eta}$ of G by*

$$\hat{\eta}(x) = \bigvee_{a \leq a_0} \{a \mid x \in \langle \eta_a \rangle\}.$$

Then $\hat{\eta} \in L(\mu, G)$ and $\hat{\eta} = \langle \eta \rangle$. Here, if η possesses sup-property, then $\hat{\eta}$ also possesses sup-property. Moreover, $\operatorname{Im}\langle \eta \rangle \subseteq \operatorname{Im} \eta$ *and $\hat{\eta}$ can also be obtained by taking supremum over* $\operatorname{Im} \eta$. *Also, $\hat{\eta}(e) = \bigvee_{x \in G} \{\eta(x)\}$.*

The following result can be obtained easily by using the above formulation of the L-subgroup generated by a given L-subset and the reasoning which is used for the corresponding result in fuzzy setting [10].

Theorem 3.2 *Let $\eta, \theta \in L^{(\mu,G)}$. Then $[\eta, \theta] = [\theta, \eta]$.*

Lemma 3.1 *Let $\eta, \theta \in \mathrm{NL}(\mu, G)$. Then for each $x, g \in G$,*

$$(\eta, \theta)(gxg^{-1}) \geq (\eta, \theta)(x) \wedge \mu(g).$$

Proof Let $x, g \in G$. If x is not a commutator, then gxg^{-1} is also not a commutator. This implies

$$(\eta, \theta)(gxg^{-1}) = \inf \eta \wedge \inf \theta,$$

and

$$(\eta, \theta)(x) \wedge \mu(g) = \inf \eta \wedge \inf \theta \wedge \mu(g) = \inf \eta \wedge \inf \theta.$$

Hence the result is true. Now for any $u \in G$, define a subset $C(u)$ of $G \times G$ as follows:

$$C(u) = \{(y, z) \in G \times G \mid u = [y, z]\}.$$

Next, suppose that x is a commutator. Then

$$\begin{aligned}
(\eta, \theta)(x) \wedge \mu(g) &= \Big\{ \bigvee_{(y,z)\in C(x)} \{\eta(y) \wedge \theta(z)\} \Big\} \wedge \mu(g) \\
&= \bigvee_{(y,z)\in C(x)} \{\eta(y) \wedge \theta(z) \wedge \mu(g)\} \\
&\qquad \text{(as } L \text{ is a completely distributive lattice)} \\
&= \bigvee_{\substack{(y^g,z^g)\in C(x^g) \\ (y,z)\in C(x)}} \{\eta(y) \wedge \theta(z) \wedge \mu(g)\} \\
&\leq \bigvee_{\substack{(y^g,z^g)\in C(x^g) \\ (y,z)\in C(x)}} \{\eta(gyg^{-1}) \wedge \theta(gzg^{-1})\} \\
&\qquad \text{(as } \eta, \theta \in \mathrm{NL}(\mu, G)) \\
&= (\eta, \theta)(gxg^{-1}).
\end{aligned}$$

In the following result, we use the normality of an L-subgroup in an L-group in the sense of Wu [21]. Therefore our method of reasoning differs from its corresponding version in fuzzy setting [10], where the parent group is an ordinary group and hence the notion of normality used is in the sense of Liu [13].

Theorem 3.3 *Let $\eta, \theta \in \mathrm{NL}(\mu, G)$. Then*

$$[\eta, \theta] \in \mathrm{NL}(\mu, G) \quad \textit{and} \quad [\eta, \theta] \subseteq \eta \cap \theta.$$

Proof Let $a_0 = \bigvee_{x\in G}\{(\eta,\theta)(x)\}$ and $x, g \in G$. Then by Theorem 3.1

$$\begin{aligned}[\eta,\theta](x) \wedge \mu(g) &= \{\bigvee_{a\le a_0}\{a \mid x \in \langle(\eta,\theta)_a\rangle\}\} \wedge \mu(g)\\ &= \bigvee_{a\le a_0}\{a \wedge \mu(g) \mid x \in \langle(\eta,\theta)_a\rangle\}\\ &\qquad\text{(as } L \text{ is a completely distributive lattice).}\end{aligned}$$

Now we claim that for $a \le a_0$,

$$\text{if } x \in \langle(\eta,\theta)_a\rangle, \text{ then } gxg^{-1} \in \langle(\eta,\theta)_c\rangle, \text{ where } c = a \wedge \mu(g).$$

Let $a \le a_0$ and $x \in \langle(\eta,\theta)_a\rangle$. Then

$$x = x_1x_2\cdots x_n, \text{ where } x_i \text{ or } x_i^{-1} \in (\eta,\theta)_a \text{ for each } i = 1,\cdots,n.$$

Thus $gxg^{-1} = (gx_1g^{-1})(gx_2g^{-1})\cdots(gx_ng^{-1})$. If $x_i \in (\eta,\theta)_a$, then by Lemma 3.1

$$(\eta,\theta)(gx_ig^{-1}) \ge (\eta,\theta)(x_i) \wedge \mu(g) \ge a \wedge \mu(g).$$

And if $x_i^{-1} \in (\eta,\theta)_a$, then repeating the above arguments we get

$$(\eta,\theta)(gx_i^{-1}g^{-1}) \ge a \wedge \mu(g).$$

This implies

$$gxg^{-1} \in \langle(\eta,\theta)_c\rangle, \quad \text{where } c = a \wedge \mu(g)$$

and hence the claim is established. Therefore in view of (1) we get

$$\begin{aligned}[\eta,\theta](x) \wedge \mu(g) &\le \bigvee_{a\le a_0}\{c \mid gxg^{-1} \in \langle(\eta,\theta)_c\rangle\}, \text{ where } c = a \wedge \mu(g)\\ &\le \bigvee_{b\le a_0}\{b \mid gxg^{-1} \in \langle(\eta,\theta)_b\rangle\}\\ &= [\eta,\theta](gxg^{-1}) \qquad \text{(by Theorem 3.1).}\end{aligned}$$

Thus $[\eta,\theta] \in \mathrm{NL}(\mu, G)$. Now, in order to prove $(\eta,\theta) \subseteq \eta \cap \theta$, let $x \in G$. If x is not a commutator, then

$$(\eta,\theta)(x) = \inf\eta \wedge \inf\theta \le \eta(x) \wedge \theta(x).$$

Hence the result is true. Suppose that $x = [y, z]$, where $y, z \in G$. Then as $\eta \in \mathrm{NL}(\mu, G)$ and $\theta \subseteq \mu$, we obtain

$$\eta(x) = \eta((yzy^{-1})z^{-1}) \ge \eta(y) \wedge \eta((zy^{-1})z^{-1}) \ge \eta(y) \wedge \mu(z) \ge \eta(y) \wedge \theta(z).$$

Similarly, as $\theta \in \mathrm{NL}(\mu, G)$ and $\eta \subseteq \mu$, we have

$$\theta(x) \geq \theta(z) \wedge \eta(y).$$

Hence,

$$\eta(y) \wedge \theta(z) \leq \eta(x) \wedge \theta(x) = \eta \cap \theta(x).$$

Consequently,

$$(\eta, \theta)(x) = \bigvee_{\substack{x=[y,z] \\ y,z \in G}} \{\eta(y) \wedge \theta(z)\} \leq \eta \cap \theta(x).$$

Thus $(\eta, \theta) \subseteq \eta \cap \theta$. Moreover as $\eta \cap \theta \in L(\mu, G)$, we have

$$[\eta, \theta] = \langle(\eta, \theta)\rangle \subseteq \eta \cap \theta.$$

The following lemma is straightforward:

Lemma 3.2 *Let $\eta, \theta \in L^{(\mu,G)}$ and $\eta \subseteq \theta$. Then $[\eta, \sigma] \subseteq [\theta, \sigma]$ for each $\sigma \in L^{(\mu,G)}$. Here also we extend another result from* [1] *to the setting of $L(\mu, G)$.*

Proposition 3.1 *Let $\eta, \theta \in L(\mu, G)$. Then $\eta \subseteq \eta \circ \theta$ and $\theta \subseteq \eta \circ \theta$ if and only if $\eta(e) = \theta(e)$.*

Next we state without proof the following results:

Proposition 3.2 *Let $\eta, \theta \in L(\mu, G)$. If $\eta(e) = \theta(e)$, then*

$$\inf \eta \circ \theta = \inf \eta \vee \inf \theta.$$

More generally, we have the following.

Proposition 3.3 *Let $\eta, \theta \in L^{(\mu,G)}$. Then*

$$\inf \eta \wedge \inf \theta \leq \inf \eta \circ \theta \leq \inf \eta \vee \inf \theta.$$

The application of the notion of infimums is exhibited in the following:

Theorem 3.4 *Let $\eta, \theta \in \mathrm{NL}(\mu, G)$ and $\sigma \in L(\mu, G)$. If either η and θ, or θ and σ have the same tails, then*

$$[\sigma \circ \eta, \theta] \subseteq [\eta, \theta] \circ [\sigma, \theta],$$

and moreover, if $\eta(e) = \sigma(e)$, then the equality holds.

Proof Let $x \in G$. If x is not a commutator and η and θ have the same tails, then

$$\begin{aligned}
[\eta,\theta]\circ[\sigma,\theta](x) &\geq [\eta,\theta](x)\wedge[\sigma,\theta](e)\\
&\geq \inf\eta\wedge\inf\theta\wedge\sigma(e)\wedge\theta(e)\\
&= \inf\theta\wedge\sigma(e)\\
&\qquad (\text{as } \inf\eta=\inf\theta \text{ and } \inf\theta\wedge\theta(e)=\inf\theta)\\
&= \inf\theta\wedge\eta(e)\wedge\sigma(e)\\
&\qquad (\text{as } \inf\theta\wedge\eta(e)=\inf\eta\wedge\eta(e)=\inf\eta=\inf\theta)\\
&= \inf\theta\wedge\sigma\circ\eta(e)\\
&\qquad (\text{as } \sigma\circ\eta(e)=\eta(e)\wedge\sigma(e))\\
&\geq \inf\theta\wedge\inf\sigma\circ\eta\\
&= (\sigma\circ\eta,\theta)(x).
\end{aligned}$$

Also if θ and σ have the same tail, then also

$$(\sigma\circ\eta,\theta)(x)\leq[\eta,\theta]\circ[\sigma,\theta](x).$$

If x is a commutator in G, then the result can be proved as in [10].

Next we provide a characterization of the notion of sup-property which lends itself more easily for applications.

Definition 3.2 *A nonempty subset X of a lattice L is said to be a supstar subset of L if every nonempty subset A of X contains its supremum. That is, if* $\sup A = a_0$*, then* $a_0 \in A$.

Proposition 3.4 *Let* $\eta \in L^{(\mu,G)}$*. Then* η *has sup-property if and only if* $\operatorname{Im}\eta$ *is a supstar subset of L.*

The above characterization also allows a generalization of the concept of sup-property to an arbitrary family of L-subsets and hence widens the scope of its applications.

Definition 3.3 *Let* $\{\eta_i\}_{i\in I}\subseteq L^{(\mu,G)}$*. Then* $\{\eta_i\}_{i\in I}$ *is said to be a supstar family if* $\bigcup\limits_{i\in I}\operatorname{Im}\eta_i$ *is a supstar subset of L.*

It is clear from the above definition and Proposition 3.4 that each member of a supstar family of L-subsets satisfies sup-property. As a particular case, we say that two L-subsets μ and η are jointly supstar if $\operatorname{Im}\mu\cup\operatorname{Im}\eta$ is a supstar subset of L.

Proposition 3.5 *Let* $\eta,\theta\in L^{(\mu,G)}$*. If* η *and* θ *are jointly supstar, then* η *and* θ *possess sup-property. Conversely, if* η *and* θ *possess sup-property, then* η *and* θ *are jointly supstar provided* $\operatorname{Im}\eta\cup\operatorname{Im}\theta$ *is a chain.*

Lemma 3.3 *Let* $\eta\in L^{(\mu,G)}$ *and* η *has sup-property. If* $a_0=\bigvee\limits_{x\in G}\{\eta(x)\}$*, then* $\langle\eta_b\rangle=\langle\eta\rangle_b$ *for each* $b\leq a_0$.

We state a similar result for the notion of strong level subsets.

Lemma 3.4 *Let $\eta \in L^{(\mu,G)}$ and $a_0 = \bigvee_{x\in G} \eta(x)$. If L is a chain, then $\langle \eta_a^> \rangle = \langle \eta \rangle_a^>$ for each $a < a_0$.*

The proof follows by the use of certain manipulations with the infimums of L-subgroups.

Theorem 3.5 *Let $\eta, \theta \in L^{(\mu,G)}$ be such that η and θ are jointly supstar. Then the commutator (η, θ) and hence the commutator L-subgroup $[\eta, \theta]$ possesses sup-property.*

The following result appears immediately.

Lemma 3.5 *Let $\eta \in L(\mu, G)$. Then $a \leq \inf \eta$ if and only if $\eta_a = G$.*

At this point, we remark that if $a \leq \inf \eta \wedge \inf \theta$, then the level subsets and $(\eta, \theta)_a$ coincide with G and hence $(\eta, \theta)_a \neq (\eta_a, \theta_a)$.

The proof of following result involves another application of infimums of L-subgroups.

Lemma 3.6 *Let $\eta, \theta \in L(\mu, G)$ be such that η and θ are jointly supstar and $a_0 = \eta(e) \wedge \theta(e)$. If $a \nleq \inf \eta \wedge \inf \theta$, then $[\eta, \theta]_a = [\eta_a, \theta_a]$ for each $a \leq a_0$.*

Let $\{\eta_i\}_{i=1}^n \subseteq L(\mu, G)$. Then we write $(\eta_1, \eta_2, \eta_3) = ((\eta_1, \eta_2), \eta_3)$ and in general

$$(\eta_1, \eta_2, \eta_3, \cdots, \eta_n) = (((\eta_1, \eta_2), \eta_3), \cdots, \eta_n).$$

Similarly, we write $[\eta_1, \eta_2, \eta_3] = [[\eta_1, \eta_2], \eta_3]$ and in general

$$[\eta_1, \eta_2, \eta_3, \cdots, \eta_n] = [[[\eta_1, \eta_2], \eta_3], \cdots, \eta_n].$$

Moreover, we point out that

$$(\eta_1, \eta_2, \eta_3, \cdots, \eta_n)(e) = \eta_1(e) \wedge \eta_2(e) \wedge \cdots \wedge \eta_n(e)$$

and

$$\inf(\eta_1, \eta_2, \eta_3, \cdots, \eta_n) = \inf \eta_1 \wedge \inf \eta_2 \wedge \cdots \wedge \inf \eta_n.$$

Also, we have

$$[\eta_1, \eta_2, \eta_3, \cdots, \eta_n](e) = \eta_1(e) \wedge \eta_2(e) \wedge \cdots \wedge \eta_n(e).$$

Theorem 3.6 *Let $\{\eta_i\}_{i=1}^n \subseteq L(\mu, G)$ be a family of L-subsets. If $a_0 = \eta_1(e) \wedge \cdots \wedge \eta_n(e)$ and $a \nleq \inf \eta_i$ for each i, then*

(i) *$[\eta_1, \eta_2, \cdots, \eta_n]_a = [(\eta_1)_a, (\eta_2)_a, \cdots, (\eta_n)_a]$ for each $a \leq a_0$, provided $\{\eta_i\}_{i=1}^n$ is a supstar family.*

(ii) *$[\eta_1, \eta_2, \cdots, \eta_n]_a^> = [(\eta_1)_a^>, (\eta_2)_a^>, \cdots, (\eta_n)_a^>]$ for each $a < a_0$, provided L is a chain.*

Corollary 3.1 *Let A and B subgroups of G. Then* $[1_A, 1_B] = 1_{[A,B]}$.

Theorem 3.7 *Let* $\{\eta_i\}_{i=1}^n \subseteq \mathrm{NL}(\mu, G)$. *Then*

$$[\eta_1, \eta_2, \cdots, \eta_n] \in \mathrm{NL}(\mu, G) \quad \textit{and} \quad [\eta_1, \eta_2, \cdots, \eta_n] \subseteq \eta_1 \cap \eta_2 \cap \cdots \cap \eta_n.$$

Proof Result can be proved by Theorem 3.4 and by an application of the principle of mathematical induction.

4. Nilpotent L-Subgroup

We start with the definition of descending central chain of an L-subgroup η of μ.

Take $Z_0(\eta) = \eta$, $Z_1(\eta) = [Z_0(\eta), \eta]$. And in general, for each i, we define $Z_i(\eta) = [Z_{i-1}(\eta), \eta]$.

Proposition 4.1 *Let* $\eta \in L(\mu, G)$. *Then for each* $i, Z_i(\eta) \subseteq Z_{i-1}(\eta)$.

Definition 4.1 *Let* $\eta \in L(\mu, G)$. *Then the chain*

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_i(\eta) \supseteq \cdots$$

of L-subgroups of μ *is called the descending central chain of* η.

It is worthwhile to note that as η is a normal L-subgroup of itself, in view of Theorem 3.7, $Z_i(\eta)$ is a normal L-subgroup of η for each i. Clearly, tip of $Z_i(\eta)$ coincides with $\eta(e)$ for each i.

Moreover, the following result for the tails of members of a descending central series can be easily established.

Proposition 4.2 *Let* $\eta \in L(\mu, G)$. *Then for each i,* $Z_i(\eta)$ *and* η *have the same tails.*

Proposition 4.3 *Let* $\eta \in L(\mu, G)$. *Then* $Z_i(\eta) \in \mathrm{NL}(\mu, G)$.

Proof Result follows by Theorem 3.7.

Definition 4.2 *Let* $\eta \in L(\mu, G)$ *with tip* a_0 *and tail* t_0. *Then the chain*

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n \supseteq \cdots$$

of L-subgroups of G is called a central chain of η *if for each i,*

$$[\eta_{i-1}, \eta] \subseteq \eta_i.$$

If $a_0 \neq t_0$ *and there exists a positive integer m such that* $\eta_m = \eta_{t_0}^{a_0}$, *where* $\eta_{t_0}^{a_0}$ *is the trivial L-subgroup of* η *with tip* a_0 *and tail* t_0, *then*

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_m = \eta_{t_0}^{a_0}$$

is known as a central series of η.

The following results are straightforward.

Proposition 4.4 *Let $\eta \in L(\mu, G)$ and*

$$\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n \supseteq \cdots$$

be a central chain of η. Then for each i, η_i and η have the same tips and also the same tails.

Lemma 4.1 *Let $\eta \in L(\mu, G)$. Then for each i and $x, y \in G$,*

$$[\eta_{i-1}, \eta] \subseteq \eta_i \text{ if and only if } \eta_{i-1(x)} \wedge \eta(y) \leq \eta_i([x, y]).$$

Proposition 4.5 *Let $\eta \in L(\mu, G)$ and $\eta = \eta_0 \supseteq \eta_1 \supseteq \cdots \supseteq \eta_n \supseteq \cdots$ be a central chain of η. Then η_i is a normal L-subgroup of η for each i.*

Definition 4.3 *Let $\eta \in L(\mu, G)$ with tip a_0 and tail t_0 and $a_0 \neq t_0$. If the descending central chain*

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_i(\eta) \supseteq \cdots$$

terminates finitely to the trivial L-subgroup $\eta_{t_0}^{a_0}$, then η is known as a nilpotent L-subgroup of μ. More precisely, η is said to be nilpotent of class c if c is the least non-negative integer such that $Z_c(\eta) = \eta_{t_0}^{a_0}$. In this case, the series $\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_c(\eta) = \eta_{t_0}^{a_0}$ is called the descending central series of η. If it is a nilpotent L-subgroup of μ, then we simply write η is nilpotent.

Next, we provide an example of a nilpotent L-subgroup of an L-group μ where neither L nor $\operatorname{Im}\eta$ is a chain.

Example 4.1 $M = \{l, f, a, b, c, d, u\}$ be a lattice given by Fig. 1. If $2 : 0 < 1$ is a chain, then

$$\begin{aligned} M \times 2 = \{&(l, 0), (f, 0), (a, 0), (b, 0), (c, 0), (d, 0), (u, 0), \\ &(l, 1), (f, 1), (a, 1), (b, 1), (c, 1), (d, 1), (u, 1)\} \end{aligned}$$

is the lattice given by Fig. 2.

Let S_4 be the group of all permutation on the set $\{1, 2, 3, 4\}$ with the identity element 'e'. Let

$$D_4^1 = \langle (24), (1234) \rangle, \quad D_4^2 = \langle (12), (1324) \rangle, \quad D_4^3 = \langle (14), (1423) \rangle$$

denote the dihedral subgroups of S_4 and $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ denote Klein-4 subgroup S_4. Define the L-subsets μ and η of S_4 as follows:

$$\mu(V_4) = (u, 1) \text{ and } \mu(S_4 \sim V_4) = (d, 1);$$

$$\eta\Big(S_4 \sim \bigcup_{i=1}^{3} D_4^i\Big) = (f, 0), \quad \eta(D_4^1 \sim V_4) = (a, 0), \quad \eta(D_4^2 \sim V_4) = (b, 0),$$

$$\eta(D_4^3 \sim V_4) = (c, 0), \quad \eta(V_4 \sim \{e\}) = (d, 0), \quad \eta(e) = (u, 0).$$

Here $A \sim B$ denotes usual set difference. Evidently, $\mu \in L(S_4)$ and $\eta \in L(\mu, S_4)$.

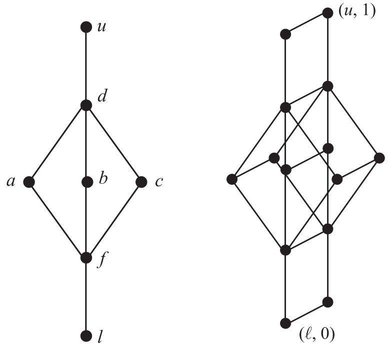

Fig. 1 Lattice M Fig. 2 Lattice $M \times 2$

Further, it is easy to verify that the commutator (η, η) is given by

$$\begin{aligned}
&(\eta, \eta)(e) = (u, 0),\\
&(\eta, \eta)((13)(24)) = (a, 0) \wedge (d, 0) = (a, 0),\\
&(\eta, \eta)((12)(34)) = (b, 0) \wedge (d, 0) = (b, 0),\\
&(\eta, \eta)((14)(23)) = (c, 0) \wedge (d, 0) = (c, 0),
\end{aligned}$$

and

$$(\eta, \eta)(S_4 \sim V_4) = (f, 0).$$

Note that $(\eta, \eta) = [\eta, \eta] \in L(\mu, S_4)$. Thus if $Z_0(\eta) = \eta$, then $Z_1(\eta) = (\eta, \eta)$.

Further, one can verify the following easily :

$$\begin{aligned}
Z_2(\eta)(e) &= (Z_1(\eta), \eta)(e) = (u, 0),\\
Z_2(\eta)(S_4 \sim \{e\}) &= (Z_1(\eta), \eta)(S_4 \sim \{e\}) = (f, 0).
\end{aligned}$$

Thus $Z_2(\eta) = \eta_{t_0}^{a_0}$ is the trivial L-subgroup of η with the tip $a_0 = \eta(e) = (u, 0)$ and the tail $t_0 = \inf \eta = (f, 0)$. This implies

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq Z_2(\eta) = \eta_{t_0}^{a_0}.$$

Hence η is a nilpotent L-subgroup of μ.

Lemma 4.2 *Let $\eta \in L(\mu, G)$ and possesses sup-property. Then for each i,*

$$\operatorname{Im} Z_i(\eta) \subseteq \operatorname{Im} \eta \cup \{\inf \eta\}.$$

Corollary 4.1 *Let $\eta \in L(\mu, G)$ and possesses sup-property. Then for each i, $Z_i(\eta)$ and η are jointly supstar.*

Lemma 4.3 *Let $\eta \in L(\mu, G)$ and possesses sup-property. Then for each $a \not\leq \inf \eta$ and $a \leq \eta(e)$, $Z_i(\eta_a) = (Z_i(\eta))_a$ for each i.*

Corollary 4.2 *Let H be a subgroup of G. Then for each i, $Z_i(1_H) = 1_{Z_i}(H)$.*

Theorem 4.1 *Let $\eta \in L(\mu, G)$ and possesses sup-property. Then η is a nilpotent L-subgroup of μ of nilpotent length at most n if and only if η_a is a nilpotent subgroup of μ_a of nilpotent length at most n for each $a \not\leq \inf \eta$ and $a \leq \eta(e)$.*

Proof $\Rightarrow$ Let $a \not\leq \inf \eta = t_0$ and $a \leq \eta(e) = a_0$. Then as η is nilpotent of class at most n, we have the descending central series terminating at the trivial L-subgroup $\eta_{t_0}^{a_0}$ of η with tip a_0 and tail t_0. That is

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq Z_2(\eta) \supseteq \cdots \supseteq Z_m(\eta) = \eta_{t_0}^{a_0},$$

where $m \leq n$. As $a \not\leq t_0$, $(\eta_{t_0}^{a_0})_a = \{e\}$. For if there exists $e \neq x \in (\eta_{t_0}^{a_0})_a$, then

$$\eta_{t_0}^{a_0}(x) = t_0 \geq a,$$

which is a contradiction as $a \not\leq t_0$. Moreover, $Z_i(\eta) \supseteq Z_{i+1}(\eta)$ so that by Proposition 2.1 $(Z_i(\eta))_a \supseteq (Z_{i+1}(\eta))_a$. Thus

$$\eta_a = (Z_0(\eta))_a \supseteq (Z_1(\eta))_a \supseteq \cdots \supseteq Z_m(\eta)_a = (\eta_{t_0}^{a_0})_a = \{e\}.$$

As η possesses sup-property, in view of Lemma 4.3,

$$Z_i(\eta_a) = (Z_i(\eta))_a \quad \text{for each } i.$$

Thus

$$\eta_a = Z_0(\eta_a) \supseteq Z_1(\eta_a) \supseteq \cdots \supseteq Z_m(\eta_a) = \{e\}.$$

Hence η_a is a nilpotent subgroup of μ_a of nilpotent length at most $m, m \leq n$.

$\Leftarrow$ As η_a is a nilpotent subgroup of μ_a for each choice of $a \not\leq t_0 = \inf \eta$ and $a \leq a_0 = \eta(e)$, there exists a positive integer $m_a \leq n$ such that

$$\eta_a = Z_0(\eta_a) \supseteq Z_1(\eta_a) \supseteq \cdots \supseteq Z_{m_a}(\eta_a) = \{e\}.$$

Let $m = \sup m_a$, where $a \in \operatorname{Im} \eta \sim \{t_0\}$. Then $m \leq n$ and

$$\eta_a = Z_0(\eta_a) \supseteq Z_1(\eta_a) \supseteq \cdots \supseteq Z_m(\eta_a) = \{e\}. \tag{1}$$

Also for each $a \leq a_0$ and $a \not\leq t_0$ $Z_i(\eta_a) = (Z_i(\eta))_a$ for each i. Hence for each $a \in \operatorname{Im} \eta \sim \{t_0\}$, we have

$$\eta_a = (Z_0(\eta))_a \supseteq \cdots \supseteq Z_{m-1}(\eta)_a \supseteq (Z_m(\eta))_a = \{e\}.$$

Next, for each $a \in \operatorname{Im}\eta \sim \{t_0\}$, $(\eta_{t_0}^{a_0})_a = \{e\}$. This implies

$$\eta_a = (Z_0(\eta))_a \supseteq \cdots \supseteq (Z_{m-1}(\eta))_a \supseteq Z_m(\eta)_a = (\eta_{t_0}^{a_0})_a.$$

Also by Proposition 4.2, $\inf Z_i(\eta) = \inf \eta$. So for each i, we have

$$(Z_i(\eta))_a = G = (\eta_{t_0}^{a_0})_a \quad \text{for} \quad a = \inf \eta = t_0.$$

Therefore for each $a \in \operatorname{Im}\eta \bigcup \{t_0\}$,

$$\eta_a = (Z_0(\eta))_a \supseteq \cdots \supseteq (Z_{m-1}(\eta))_a \supseteq (Z_m(\eta))_a = (\eta_{t_0}^{a_0})_a.$$

Moreover in view of Lemma 4.2, we have

$$\operatorname{Im} Z_i(\eta) \subseteq \operatorname{Im}\eta \cup \{t_0\} \quad \text{for each } i.$$

So that

$$(Z_{i-1}(\eta))_a \supseteq (Z_i(\eta))_a \quad \text{for each } a \in \operatorname{Im} Z_i(\eta).$$

Thus by Proposition 2.1, $Z_{i-1}(\eta) \supseteq Z_i(\eta)$ for each i. We claim that

$$Z_m(\eta) = \eta_{t_0}^{a_0}.$$

Note that

$$(Z_m(\eta))_a = \{e\} = (\eta_{t_0}^{a_0})_a \ \text{ for each } a \in \operatorname{Im}\eta \sim \{t_0\},$$
$$(Z_m(\eta))_a = G = (\eta_{t_0}^{a_0})_a \quad \text{for } a = \inf \eta = t_0.$$

Moreover, $\operatorname{Im} Z_m(\eta) \subseteq \operatorname{Im}\eta \bigcup \{t_0\}$. Hence by Proposition 2.1, $Z_m(\eta) = \eta_{t_0}^{a_0}$. Consequently,

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_{m-1}(\eta) \supseteq Z_m(\eta) = \eta_{t_0}^{a_0}.$$

This establishes that η is a nilpotent L-subgroup of μ of nilpotent length at most n.

The following result is stated without proof.

Theorem 4.2 *Let $\eta \in L(\mu, G)$ and L be a chain. Then η is a nilpotent L-subgroup of μ of nilpotent length at most n if and only if $\eta_a^>$ is a nilpotent subgroup of $\mu_a^>$ of nilpotent length at most n, where $a < \eta(e)$.*

Corollary 4.3 *Let H be a subgroup of a group G. Then H is a nilpotent subgroup of G if and only if 1_H is a nilpotent L-subgroup of 1_G.*

Proposition 4.6 *Let $\eta \in L(\mu, G)$ be a proper L-subgroup of μ. Then η is nilpotent if and only if η has a central series.*

Theorem 4.3 *Let $\eta \in L(\mu, G)$ and θ be a proper L-subgroup of having the common tail t_0. If η is nilpotent, then θ is also nilpotent.*

5. Conclusion

It is after the emergence of metatheorem by Prof. Tom Head that the work in various disciplines of fuzzy algebra came to a standstill due to the fact that the most of the notions and concepts considered in these disciplines are generically defined. Therefore the extension of the results in the fuzzy setting of classical algebra become simple instances of application of metatheorem and subdirect product theorem. In order to avoid this situation, we suggest the researchers pursuing studies in these areas to switch over to L-setting where the metatheorem is not applicable. Moreover, we also suggest to investigate properties of L-subalgebras of an L-algebra instead of a classical algebra. For such studies, the infimums of L-subalgebras, as is demonstrated in this paper, may play a significant role.

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